Solving a Cubic Equation

What is a Cubic Equation?

A cubic equation is an equation of form

$$ax^3+bx^2+cx+d=0$$

where

$$a\neq 0$$

The Formula

The cubic formula dates back to the 16th century when scholars challenged each othe to mathematical duels.

So your weapons had to be kept secret.

Scipione del Ferro, a professor in the university of bologna knew a formula to solve equations of form

$$x^3 + cx = d$$

Where c and d are positive

(those were the rules)

$$x=\sqrt[3]{\frac{d}{2}+\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}+\sqrt[3]{\frac{d}{2}-\sqrt{\frac{d^2}{4}+\frac{c^3}{27}}}$$

Proof: cube both sides.

Del Ferro taught the formula to his student Antonio Fior, who in turn challenged Tartaglia to a duel with 30 depressed cubic equations, that Tartaglia solved in 2 hours.

To make a long story short, in 1539 Tartaglia shared the formula with Cardano, and Cardano found that any cubic equation can be turned into a depressed one using the substitution:

$$x = t - \frac{b}{3a}$$

The full article can be found at https://pnqk.me/gv6h6f

Let Us Find it Ourselves

We have seen from del Ferro’s formula that the solution of a depressed cubic equation is the sum of two cube roots. Let us call the *u* and *v*.

Now, be our equation:

$$ax^3+bx^2+cx+d=0$$

Divide by *a:*

$$x^3+\frac{bx^2}{a}+\frac{cx}{a}+\frac{d}{a}=0 \quad (1)$$

Now, the expansion of $\left(x+\frac{b}{3a}\right)^3$ is:

$$\left(x+\frac{b}{3a}\right)^3 = x^3 + \frac{bx^2}{a} + \frac{b^2x}{3a^2} + \frac{b^3}{27a^3}$$

So, we can rewrite equation (1) as:

$$\left(x+\frac{b}{3a}\right)^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)x + \frac{d}{a} - \frac{b^3}{27a^3} = 0$$

And then as:

$$\left(x+\frac{b}{3a}\right)^3+\left(\frac{c}{a}-\frac{b^2}{3a^2}\right)\left(x+\frac{b}{3a}\right)+\frac{d}{a}-\frac{bc}{3a^2}+\frac{b^3}{27a^3}=0$$

That’s a depressed cubic equation!

Now, let:

$$p=\frac{c}{a}-\frac{b^2}{3a^2},\quad q=\frac{d}{a}-\frac{bc}{3a^2}+\frac{b^3}{27a^3}$$

So, we have:

$$\left(x+\frac{b}{3a}\right)^3=-p\left(x+\frac{b}{3a}\right)-q \quad (3)$$

Now, let us find pairs of u,v, such that $x+\frac{b}{3a}=u+v$

is always a solution. Then:

$$(u+v)^3=3uv(u+v)+u^3+v^3$$

By equating the coefficients with thoes of (3), we get the system:

$$\begin{cases} 3uv=-p \\ u^3+v^3=-q \end{cases}$$

Let us cube both sides of the first equation to solve for u^3 and v^3:

$$\begin{cases} 27u^3v^3 = -p^3 \\ u^3 + v^3 = -q \end{cases}$$

From the system above:

$$u^3(-q - u^3) = -\frac{p^3}{27}$$

$$\Downarrow$$

$$u^6 + qu^3 - \frac{p^3}{27} = 0$$

From the quadratic formula:

$$u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Without loss of generality:

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \qquad v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

I’m not in a hurry to take cube roots because cubing both sides of an equation creates extraneous solutions.

Instead, I will take:

$$u=\left(-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}$$

If u=0:

$$v=\left(-\frac{q}{2}-\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}\right)^{\frac{1}{3}}$$

Otherwise:

$$v=\frac{uv}{u}=-\frac{p}{3u}$$

Note: If you take a cube root of a complex number, use the polar form and de Moivre’s theorem, or you will need a system of cubic equations to get the roots.

Let us take now, $w = \frac{-1+\sqrt{3}i}{2}$

Then, the solutions are:

$$x_1 = u + v - \frac{b}{3a}$$

$$x_2 = wu + w^2 v - \frac{b}{3a}$$

$$x_3 = w^2 u + w v - \frac{b}{3a}$$

¿

Cubic Discriminant

When the coefficients of a cubic equation are real, a cubic discriminant is defined as follows:

$$\Delta = b^2 c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2 + 18abcd$$

- If $\Delta > 0$ there are three distinct real roots
- If $\Delta = 0$ there is a repeated real root and all roots are real.
- If $\Delta < 0$ there is only one real root.

And if there are three real roots, you can use trigonometry and hide the use of complex numbers.