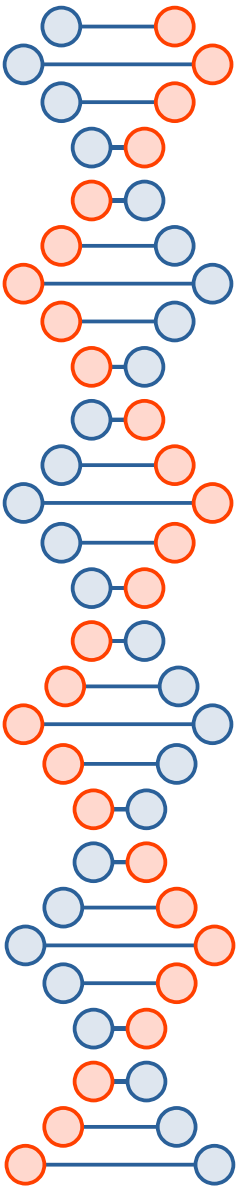
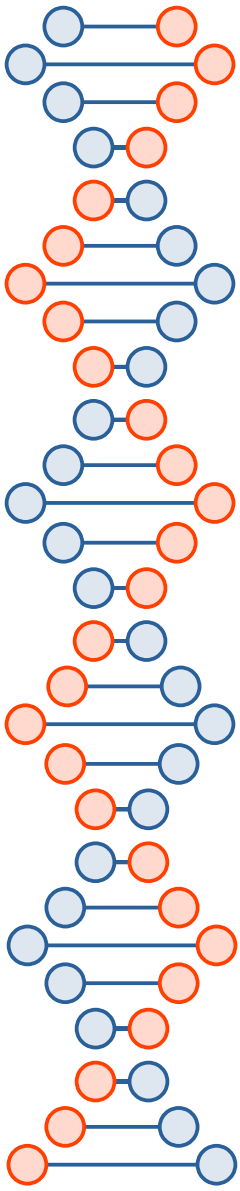


Solving a Cubic Equation





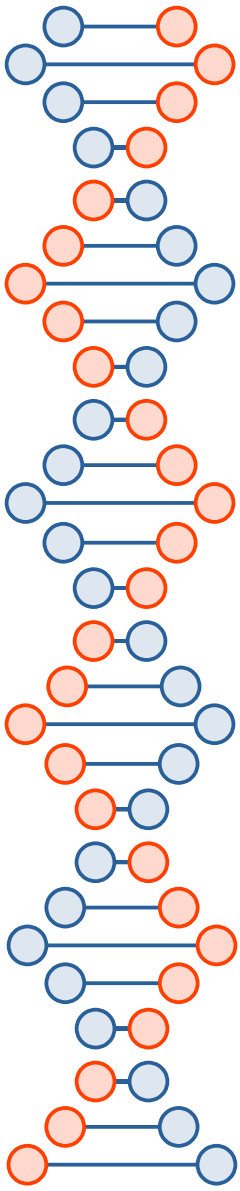
What is a Cubic Equation?

A cubic equation is an equation of form

$$ax^3 + bx^2 + cx + d = 0$$

where

$$a \neq 0$$



The Formula

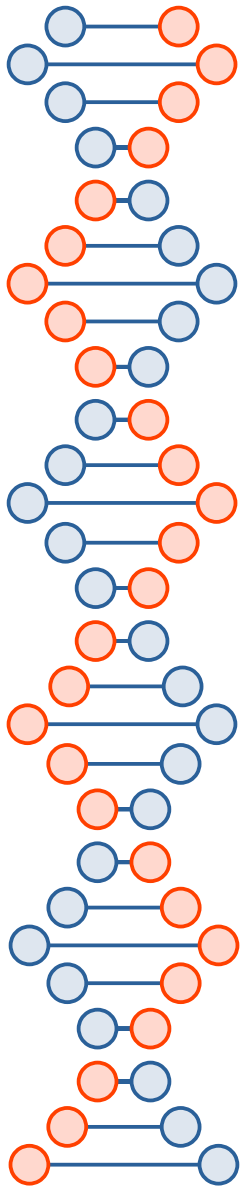
The cubic formula dates back to the 16th century when scholars challenged each other to mathematical duels.

So your weapons had to be kept secret.

Scipione del Ferro, a professor in the university of Bologna knew a formula to solve equations of form

$$x^3 + cx = d$$

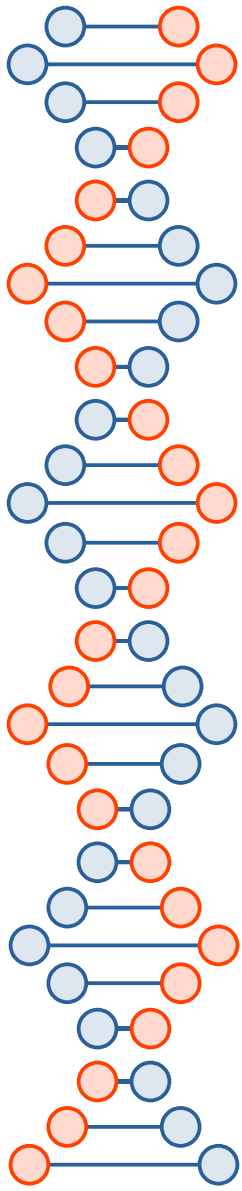
Where c and d are positive
(those were the rules)



$$x = \sqrt[3]{\frac{d}{2} + \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}} + \sqrt[3]{\frac{d}{2} - \sqrt{\frac{d^2}{4} + \frac{c^3}{27}}}$$

Proof: cube both sides.

Del Ferro taught the formula to his student Antonio Fior, who in turn challenged Tartaglia to a duel with 30 depressed cubic equations, that Tartaglia solved in 2 hours.



To make a long story short, in 1539 Tartaglia shared the formula with Cardano, and Cardano found that any cubic equation can be turned into a depressed one using the substitution:

$$x = t - \frac{b}{3a}$$

The full article can be found at <https://pnqk.me/gv6h6f>

Let Us Find it Ourselves

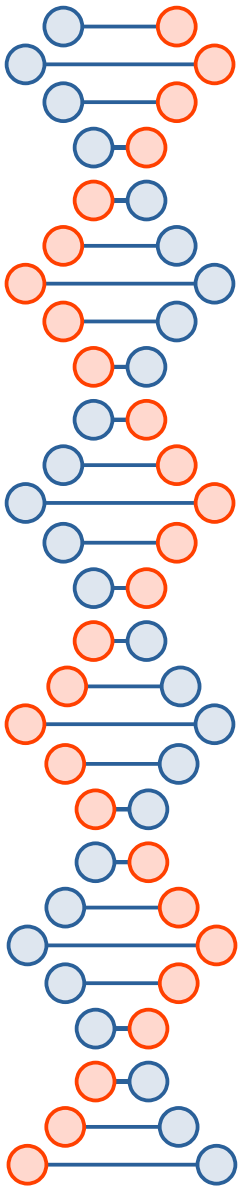
We have seen from del Ferro's formula that the solution of a depressed cubic equation is the sum of two cube roots. Let us call the u and v .

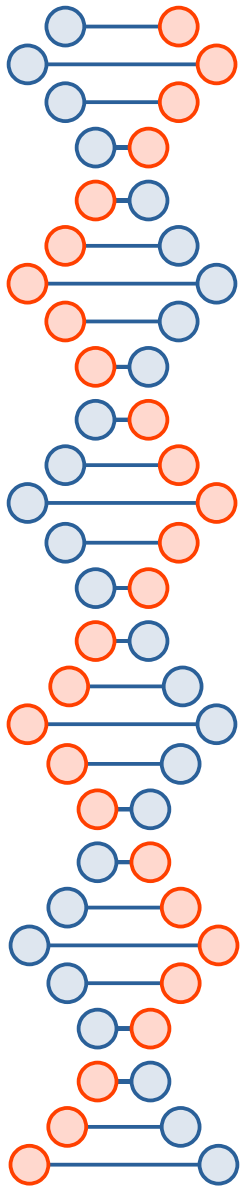
Now, be our equation:

$$ax^3 + bx^2 + cx + d = 0$$

Divide by a :

$$x^3 + \frac{bx^2}{a} + \frac{cx}{a} + \frac{d}{a} = 0 \quad (1)$$



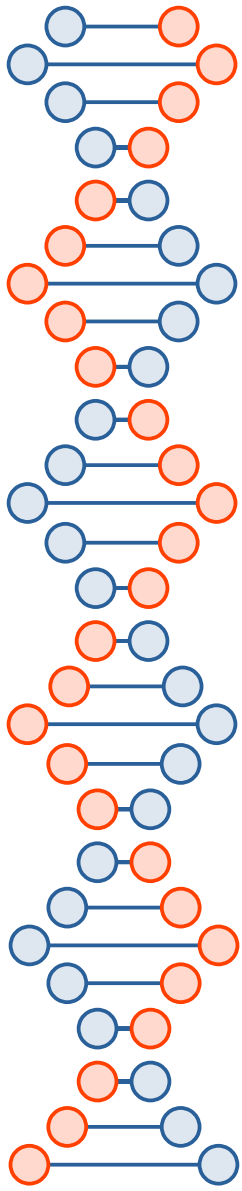


Now, the expansion of $\left(x + \frac{b}{3a}\right)^3$ is:

$$\left(x + \frac{b}{3a}\right)^3 = x^3 + \frac{bx^2}{a} + \frac{b^2x}{3a^2} + \frac{b^3}{27a^3}$$

So, we can rewrite equation (1) as:

$$\left(x + \frac{b}{3a}\right)^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)x + \frac{d}{a} - \frac{b^3}{27a^3} = 0$$



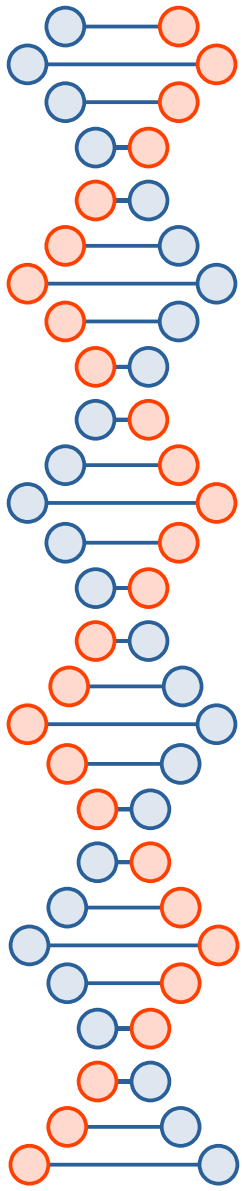
And then as:

$$\left(x + \frac{b}{3a}\right)^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)\left(x + \frac{b}{3a}\right) + \frac{d}{a} - \frac{bc}{3a^2} + \frac{b^3}{27a^3} = 0$$

That's a depressed cubic equation!

Now, let:

$$p = \frac{c}{a} - \frac{b^2}{3a^2}, \quad q = \frac{d}{a} - \frac{bc}{3a^2} + \frac{b^3}{27a^3}$$



So, we have:

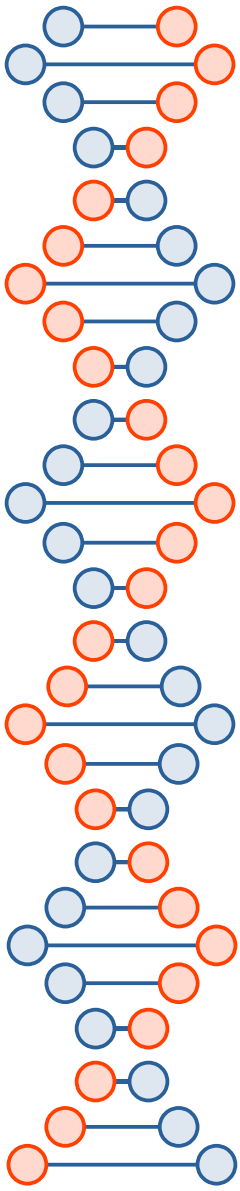
$$\left(x + \frac{b}{3a}\right)^3 = -p\left(x + \frac{b}{3a}\right) - q \quad (3)$$

Now, let us find pairs of u, v , such that $x + \frac{b}{3a} = u + v$ is always a solution. Then:

$$(u+v)^3 = 3uv(u+v) + u^3 + v^3$$

By equating the coefficients with those of (3), we get the system:

$$\begin{cases} 3uv = -p \\ u^3 + v^3 = -q \end{cases}$$



Let us cube both sides of the first equation to solve for u^3 and v^3 :

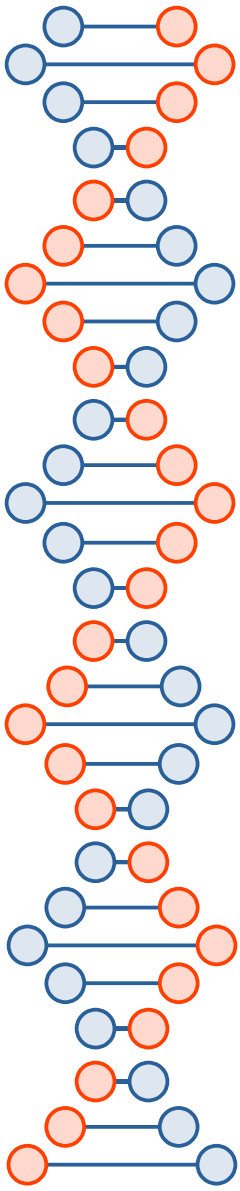
$$\begin{cases} 27u^3v^3 = -p^3 \\ u^3 + v^3 = -q \end{cases}$$

From the system above:

$$u^3(-q - u^3) = -\frac{p^3}{27}$$



$$u^6 + qu^3 - \frac{p^3}{27} = 0$$



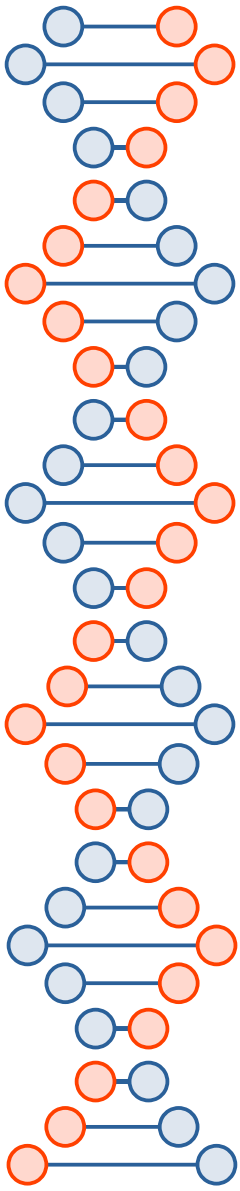
From the quadratic formula:

$$u^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2} = -\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

Without loss of generality:

$$u^3 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}, \quad v^3 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

I'm not in a hurry to take cube roots because cubing both sides of an equation creates extraneous solutions.



Instead, I will take:

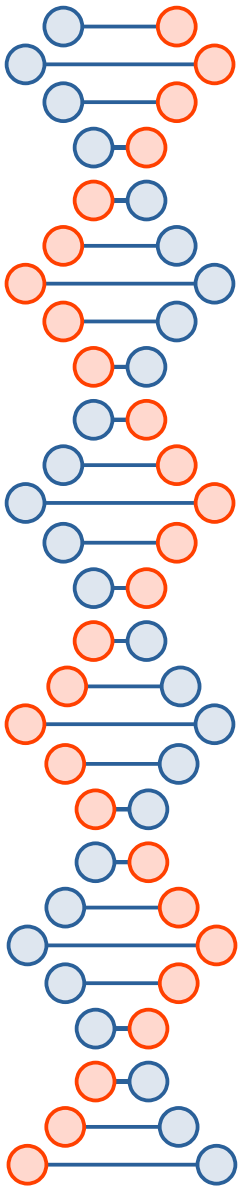
$$u = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}}$$

If $u=0$:

$$v = \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \right)^{\frac{1}{3}}$$

Otherwise:

$$v = \frac{uv}{u} = -\frac{p}{3u}$$



Note: If you take a cube root of a complex number, use the polar form and de Moivre's theorem, or you will need a system of cubic equations to get the roots.

Let us take now, $w = \frac{-1 + \sqrt{3}i}{2}$

Then, the solutions are:

$$x_1 = u + v - \frac{b}{3a}$$

$$x_2 = wu + w^2 v - \frac{b}{3a}$$

$$x_3 = w^2 u + w v - \frac{b}{3a}$$

i

Cubic Discriminant

When the coefficients of a cubic equation are real, a cubic discriminant is defined as follows:

$$\Delta = b^2 c^2 - 4ac^3 - 4b^3 d - 27a^2 d^2 + 18abcd$$

- If $\Delta > 0$ there are three distinct real roots
- If $\Delta = 0$ there is a repeated real root and all roots are real.
- If $\Delta < 0$ there is only one real root.

And if there are three real roots, you can use trigonometry and hide the use of complex numbers.