TAMI π
A Light Review on Lie Groups

Definition, Types, and Connections to Algebraic Functions

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Overview

Topics Covered:

- Introduction to Lie Groups
- Formal Definitions and Key Concepts
- Types of Lie Groups
- Examples of Lie Groups
- Non-Examples of Lie Groups
- Algebraic Functions as Lie Groups?

Introduction

Groups

Differentiable Manifolds

Differentiable Manifolds

A differentiable manifold is a type of mathematical space that, at every point, resembles Euclidean space and allows for calculus operations.

This structure enables the definition of smooth functions and differentiation.

Groups

A group is a mathematical structure consisting of a set of elements equipped with an operation that combines any two elements to form another element, satisfying four properties:

Closure

Associativity

∃ an identity element

∃ inverse elements.

What is a Lie Group?

Definition: A group that is also a smooth differentiable manifold, where the group operations (multiplication and inversion) are smooth.

Historical Background: Named after Sophus Lie.

Importance: Used in mathematics and physics for describing continuous symmetries.

Key Characteristics

Differentiable Manifold: Locally resembles Euclidean space.

Smooth Operations: Functions describing group operations are differentiable.

Example: The group of rotations in 3D space, SO(3).

Formal Definition

Group Structure: Set G with operations satisfying:

- Associativity
- Identity element
- Inverse elements

Manifold Structure: Compatible with group operations.

Types of Lie Groups

Compact Lie Groups: Closed and bounded.

Non-Compact Lie Groups: Not bounded.

Connected Lie Groups: Path-connected.

Disconnected Lie Groups: Multiple components.

Examples of Lie Groups

SO(2): Rotations in 2D.

SU(2): Special unitary group in 2D complex space.

GL(n, ℝ): General linear group over real numbers.

Non-Examples of Lie Groups

Integers under Addition: $\mathbb{Z}$ is a discrete group, not a manifold.

Rational Numbers under Multiplication: $\mathbb{Q}^+$ lacks smooth manifold structure.

Discrete Symmetry Groups: Like permutation groups, which are not continuous.

Compact Lie Groups

Definition: Lie groups that are compact as topological spaces.

Examples: SO(n), SU(n), U(n).

Applications: Symmetries in physics, such as SO(3) in quantum mechanics.

Non-Compact Lie Groups

Definition: Lie groups that are not compact.

Examples: $GL(n, \mathbb{R})$, $SL(2, \mathbb{R})$.

Applications: Relativity and transformations in physics.

Connected and Disconnected Lie Groups

Connected Groups: Can be continuously transformed within the group.

Disconnected Groups: Multiple components that cannot be connected.

Examples: SO(3) is connected; O(3) is disconnected.

Lie Algebras

Definition: Tangent space at the identity element of a Lie group forms a Lie algebra.

Structure: Related to infinitesimal transformations.

Example: Lie algebra of SO(3) is $\mathfrak{so}(3)$.

Connection to Algebraic Functions

Exploring if algebraic operations form Lie groups:

Addition: $\mathbb{R}$ under addition is a Lie group.

Multiplication: $\mathbb{R}^+$ under multiplication is a Lie group.

Analysis of Algebraic Structures

Addition and Multiplication:

- Smooth structures
- Identity and inverse elements

Polynomial Functions: Generally, do not form Lie groups.

Example Analysis

Example 1: $\mathbb{R}$ under addition

- Associativity, identity (0), inverse (negation).

Example 2: $\mathbb{R}^+$ under multiplication

- Associativity, identity (1), inverse (reciprocal).

Applications of Lie Groups

Physics: Symmetries and conservation laws.

Engineering: Robotics and control theory.

Mathematics: Differential equations and topology.

Summary

Lie groups combine algebraic and geometric properties.

Importance: Essential in theoretical frameworks.

Questions and complains