The Stone Theorem of Single Parameter Unitary Group

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The Schrödinger equation

Time independent

$$\hat{\mathrm{H}}\,|\Psi\rangle = E|\Psi\rangle$$

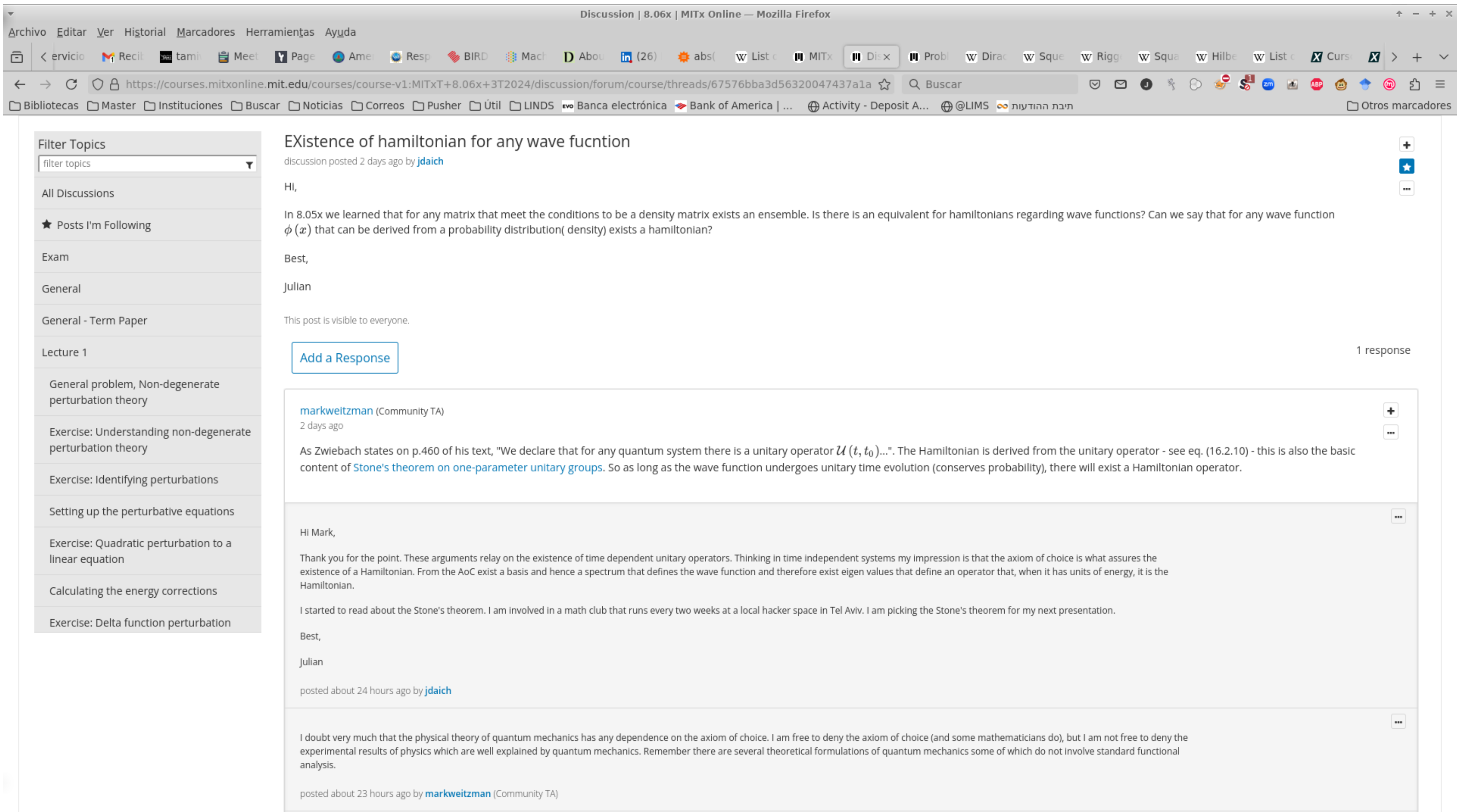
Time dependent

$$i\hbar\frac{d}{dt}|\Psi(t)\rangle = \hat{H}|\Psi(t)\rangle$$

One dimensional potential example

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t).$$

The Stone Theorem of Single Parameter Unitary Group



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General problem, Non-degenerate perturbation theory
Exercise: Understanding non-degenerate perturbation theory
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Setting up the perturbative equations
Exercise: Quadratic perturbation to a linear equation
Calculating the energy corrections
Exercise: Delta function perturbation

EXistence of hamiltonian for any wave fucntion

discussion posted 2 days ago by jdaich

Hi,

In 8.05x we learned that for any matrix that meet the conditions to be a density matrix exists an ensemble. Is there is an equivalent for hamiltonians regarding wave functions? Can we say that for any wave function $\phi(x)$ that can be derived from a probability distribution(density) exists a hamiltonian?

Best,

Julian

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1 response

markweitzman (Community TA)
2 days ago

As Zwiebach states on p.460 of his text, "We declare that for any quantum system there is a unitary operator $\mathcal{U}(t, t_0)$...". The Hamiltonian is derived from the unitary operator - see eq. (16.2.10) - this is also the basic content of Stone's theorem on one-parameter unitary groups. So as long as the wave function undergoes unitary time evolution (conserves probability), there will exist a Hamiltonian operator.

Hi Mark,

Thank you for the point. These arguments relay on the existence of time dependent unitary operators. Thinking in time independent systems my impression is that the axiom of choice is what assures the existence of a Hamiltonian. From the AoC exist a basis and hence a spectrum that defines the wave function and therefore exist eigen values that define an operator that, when it has units of energy, it is the Hamiltonian.

I started to read about the Stone's theorem. I am involved in a math club that runs every two weeks at a local hacker space in Tel Aviv. I am picking the Stone's theorem for my next presentation.

Best,

Julian

posted about 24 hours ago by jdaich

I doubt very much that the physical theory of quantum mechanics has any dependence on the axiom of choice. I am free to deny the axiom of choice (and some mathematicians do), but I am not free to deny the experimental results of physics which are well explained by quantum mechanics. Remember there are several theoretical formulations of quantum mechanics some of which do not involve standard functional analysis.

posted about 23 hours ago by markweitzman (Community TA)

Background

Definition:

A group of operators $(U_t)_{t\in\mathbb{R}}$ where $\forall t \in \mathbb{R}, U_t U_t^T = I$, are unitary and strongly continuous

$$\forall t_0 \in \mathbb{R}, \psi \in \mathcal{H} : \qquad \lim_{t\to t_0} U_t(\psi) = U_{t_0}(\psi)$$

Key Property: Homomorphism, $U_{t+s} = U_t U_s$ and $U_0 = I$

Definition:

An operator A such that $A = A^*$ is called Hermitian if it is equal to its adjoint.

Key property A generates a strongly continuous unitary group.

The Stone theorem connects unitary groups to self-adjoint operators.

Application in Functional Analysis and Quantum Mechanics.

Stone's Theorem

Theorem Statement:

Let's define $U_t = U(t)$ and $U_t(\psi) = U(t)\psi$

Every strongly continuous one-parameter unitary group $(U_t)_{t\in\mathbb{R}}$ is generated by a unique self-adjoint operator A.

$U(t) = e^{iAt}$

, where is the generator.

$U(t)$ satisfies the Schrödinger equation:

$$i\frac{d}{dt}U(t)\psi = AU(t)\psi$$

Proof Outline

Define the generator $A : \mathcal{D}_A \to \mathcal{H}$ using strong continuity

$$\mathcal{D}_A = \left\{ \psi \in \mathcal{H} \,\middle|\, \lim_{\varepsilon \to 0} \frac{-i}{\varepsilon} \left(U_\varepsilon(\psi) - \psi \right) \text{ exists} \right\}$$

$\lim_{t \to 0} \|U_t \psi - \psi\| = 0, \quad \forall \psi \in \mathcal{H}$, $\forall t \in \mathbb{R} :$ $\qquad U_t = e^{itA}$

Show $U(t) = e^{iAt}$ satisfies unitary properties

$$\forall t \in \mathbb{R}, U_t U_t^T = U(t) U(t)^* = I$$

Uniqueness of A by spectral decomposition.

$$U(t) = e^{itA} = \int_{-\infty}^{\infty} e^{it\lambda} \, dE_A(\lambda)$$

Proof- One side

Define the generator $A : \mathcal{D}_A \to \mathcal{H}$ using strong continuity

$$A(\psi) \overset{\text{def}}{=} i \lim_{t\to 0} \frac{U(t)\psi - \psi}{t}.$$

If $U(t) = e^{iAt}$ we have to show hermiticity $A = A^*$

$$\langle A\phi \mid \psi \rangle = -i \lim_{t\to 0} \frac{\langle U(t)\phi \mid \psi\rangle - \langle \phi \mid \psi \rangle}{t}.$$

$$U(t)U(t)^* = U(t)U(-t) \Rightarrow \langle U(t)\phi \mid \psi \rangle = \langle \phi \mid U(-t)\psi \rangle.$$

$$\langle A\phi \mid \psi \rangle = \langle \phi \mid A\psi \rangle.$$

From the spectral decomposition uniqueness.

$U(t) = e^{itA} \overset{\text{def}}{=} \int_{-\infty}^{\infty} e^{it\lambda}\, dE_A(\lambda)$, E_λ spectral measure of A

Spectral theorem representation

From the spectral decomposition

$$U(t) = e^{itA} \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} e^{it\lambda}\, dE_A(\lambda),$$ E_λ a spectral measure

If $U(t) = e^{itB} = \int_{-\infty}^{\infty} e^{it\lambda}\, dE_B(\lambda)$ then

$$\forall t\ U(t) = e^{itA} = e^{itB} \Rightarrow A = B$$

Proof- Second side

Define the generator $A : \mathcal{D}_A \to \mathcal{H}$ using strong continuity

$$A(\psi) \stackrel{\text{def}}{=} i \lim_{t \to 0} \frac{U(t)\psi - \psi}{t}, \quad \forall \psi \in \mathcal{D}(A).$$

From the spectral decomposition uniqueness.

$$A\psi = \int_{-\infty}^{\infty} \lambda \, dE_A(\lambda)\psi \Rightarrow f(A)\psi = \int_{-\infty}^{\infty} f(\lambda) \, dE_A(\lambda)\psi$$

$$\Rightarrow e^{itA} = \int_{-\infty}^{\infty} e^{it\lambda} \, dE_A(\lambda) = U(t)$$

Proof- Second side

If $U(t) = e^{iAt}$ and $A = A^*$

$$U(0) = e^{i(0)A} = \int_{-\infty}^{\infty} 1\, dE_A(\lambda) = I.$$

$$U(t+s) = e^{iA(t+s)} = U(t)U(s).$$

$$U(t)^* = \left(\int_{-\infty}^{\infty} e^{it\lambda}\, dE_A(\lambda) \right)^* = U(-t).$$

Since A is strongly continuous

$$A\psi = i \lim_{t \to 0} \frac{\left(\int_{-\infty}^{\infty} e^{it\lambda}\, dE_A(\lambda) \right) \psi - \psi}{t} = \int_{-\infty}^{\infty} i\lambda\, dE_A(\lambda)\psi.$$

Then A is an infinitesimal generator

Applications

Quantum Mechanics: Evolution of quantum states.

$$\frac{d}{dt}A_{\mathrm{H}}(t) = \frac{i}{\hbar}[H_{\mathrm{H}}(t), A_{\mathrm{H}}(t)] + \left(\frac{\partial A_{\mathrm{S}}}{\partial t}\right)_{\mathrm{H}},$$

$$U(t) = e^{-\frac{i}{\hbar}tH_{\mathrm{S}}} \Rightarrow \langle A\rangle_t = \langle\psi(0)|e^{\frac{i}{\hbar}tH}A_{\mathrm{S}}(t)e^{-\frac{i}{\hbar}tH}|\psi(0)\rangle$$

Differential Equations: Solving time-dependent PDEs.

$$\frac{\partial u}{\partial t} = i\frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in \mathbb{R}\times\mathbb{R}, \quad u(x,0) = f(x).$$

$$\frac{du}{dt} = iAu,$$

Conclusion

Stone's theorem is fundamental in understanding dynamics in quantum systems and operator theory.

It bridges abstract mathematics and physical applications.

Let’s discuss about it

Thanks!!